“The Axial Anomaly in D=3+1 Light-Cone QED”

hep-ph/0207190 with R. P. Woodard

Light-Cone 2002
Los Alamos National Laboratory, NM
August 8, 2002

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A Brief Synopsis

- Turn on $\vec{E}$ and $\vec{B}$ fields in the $+z$ direction.
- Solve the light-cone Dirac equation exactly regarding E and B as external $\mathbb{C}$-number fields.
- Compute the probability of creating definitive states of electron-positron pairs.
- Compute the VEVs of the light-cone vector current operators and confirm current conservation.
- Compute the VEVs of the light-cone axial current and pseudoscalar operators and confirm the axial vector anomaly.
- Show the absolute necessity of including initial value data from both $x^+ = 0$ AND $x^- = -L$

The Model and its Solution

- L.C. conventions,

$$
\begin{aligned}
x^{\pm} &\equiv \frac{1}{\sqrt{2}}(x^0 \pm x^3) \qquad , \qquad x^{\perp} \equiv (x^1, x^2) \\
a^{\mu} b_{\mu} &= a^+ b^- + a^- b^+ - a^{\perp} b^{\perp}
\end{aligned}
$$

- L.C. Dirac spinors

$$
\begin{aligned}
\psi_{\pm} &= P_{\pm}\Psi \qquad , \qquad \Psi = \psi_+ + \psi_- \\
P_{\pm} &\equiv \frac{1}{2}\gamma^{\mp}\gamma^{\pm} = \frac{1}{2}(I \pm \gamma^0\gamma^3)
\end{aligned}
$$

- Background

$$
\vec{E}(x) = E(x^+)\hat{x}_3 \qquad , \qquad \vec{B}(x) = B\hat{x}_3
$$

- Gauge and surface choices

$$
\begin{aligned}
A_+(x^+, x^-, x^{\perp}) &= 0 \\
A_-(0, x^-, x^{\perp}) &= 0 \\
A_1(0, 0, x^{\perp}) &= -A_2(0, 0, x^{\perp})
\end{aligned}
$$

- L.C. vector potential

$$
A_-(x^+) = -\int_0^{x^+} dy E(y) \quad , \quad A_{\perp}(x^{\perp}) = \frac{B}{2}(x^2\hat{x}_1 - x^1\hat{x}_2)
$$

- L.C. Dirac equation

$$[i\gamma^+\partial_+ + i\gamma^-(\partial_- + ieA_-) + i\gamma^\perp \cdot \mathcal{D}_\perp - m]\Psi(x) = 0$$

$$\begin{aligned} i\partial_+\psi_+(x) &= \frac{1}{2}(m + i\gamma^\perp \cdot \mathcal{D}_\perp)\gamma^-\psi_- \\ (i\partial_- - eA_-)\psi_-(x) &= \frac{1}{2}(m + i\gamma^\perp \cdot \mathcal{D}_\perp)\gamma^+\psi_+ \end{aligned}$$

- Solution

$$\psi_+(x^+, x^-, x^\perp) = \int_{-L}^{\infty} dv \int_{-\infty}^{+\infty} \frac{dk^+}{2\pi} e^{i(k^+ + i/L)(v - x^-)}$$
$$\times \Big\{ \mathcal{U}\left(x^\perp, \tau_+\right) \psi_+(0, v, x^\perp) - \frac{i}{2}(m + i\gamma^\perp \cdot \mathcal{D}_\perp)$$
$$\int_0^{x^+} du e^{-ieA_-(u)(v+L)} \mathcal{U}\left(x^\perp, \tau_-\right) \gamma^- \psi_-(u, -L, x^\perp) \Big\},$$

$$\psi_- = i\gamma^+(m + i\gamma^\perp \cdot \mathcal{D}_\perp)^{-1}\partial_+\psi_+$$

- Definitions

$$\begin{aligned} \mathcal{U}(x^\perp, \tau) &\equiv e^{-i\mathcal{H}[eA_\perp(x_\perp)]\tau}, \\ \mathcal{H}[eA_\perp(x_\perp)] &\equiv \frac{1}{2}(m + i\gamma^\perp \cdot \mathcal{D}_\perp)(m - i\gamma^\perp \cdot \mathcal{D}_\perp) \\ \tau_+ &\equiv \int_0^{x^+} \frac{dy}{k^+ - eA_-(y) + i/L} \end{aligned}$$

$$\tau_- \equiv \int_u^{x^+} \frac{dy}{k^+ - eA_-(y) - i/L}$$

- $\mathcal{U}$ as a kernel for the first quantized Hamiltonian, $\mathcal{H}$

$$\begin{aligned}
\mathcal{U}(x^\perp,\tau)f(x^\perp) &= \int d^2y^\perp K(x^\perp,y^\perp;\tau)f(y^\perp) \\
\mathcal{K}(x^\perp,y^\perp;\tau) &= \frac{\beta e^{-\frac{i}{2}(m^2+2\beta\Sigma^3)\tau}}{2i\pi\sin(\beta\tau)} \\
&\quad \times e^{\frac{i\beta}{2}\cot(\beta\tau)(x^\perp-y^\perp)^2 - iex^\perp\cdot A_\perp(y^\perp)}
\end{aligned}$$

$$\beta \equiv \frac{|e|B}{2} \qquad , \qquad \Sigma^3 \equiv \frac{i}{2}[\gamma^1,\gamma^2]$$

- The diagonal representation of $\mathcal{H}$ in terms of Landau levels

$$\begin{aligned}
\mathcal{H} &= \frac{1}{2}m^2 + (2a_-^\dagger a_- + 1 + \Sigma^3)\beta \\
W_{n_\pm}(x^\perp) &\equiv \frac{(a_+^\dagger)^{n_+}}{\sqrt{n_+!}}\frac{(a_-^\dagger)^{n_-}}{\sqrt{n_-!}}\sqrt{\frac{\beta}{\pi}}e^{-\frac{\beta}{2}\|x^\perp\|^2} \\
\psi_\pm(x^+,x^-,n_\pm) &= \int d^2x^\perp W_{n_\pm}^*(x^\perp)\psi_\pm(x^+,x^-,x^\perp) \\
&\equiv \langle\!\langle W_{n_\pm}|\psi_\pm(x^+,x^-)\rangle\!\rangle
\end{aligned}$$

- Eigenoperator of L.C. Hamiltonian

$$
\begin{aligned}
&-i\partial_+\widetilde{\psi}_+(x^+,k^+,n_\pm) = \\
&\qquad -\frac{\frac{1}{2}m^2+(2n_-+1+\Sigma^3)\beta}{k^+-eA_-(x^+)+i/L}\widetilde{\psi}_+(x^+,k^+,n_\pm) \\
&\qquad -\frac{e^{-i(k^++i/L)}}{k^+-eA_-(x^+)+i/L}\frac{i}{2} \\
&\qquad \times\langle\!\langle W_{n_\pm}|(m+i\gamma^\perp\cdot\mathcal{D}_\perp)\gamma^-|\psi_-(x^+,-L)\rangle\!\rangle
\end{aligned}
$$

- Pair creation probability for $x^+ > X(k^+)$

$$
\begin{aligned}
&\lim_{L\to\infty}\frac{1}{\sqrt{2}}\langle\Omega|\widetilde{\psi}_+^\dagger(x^+,q^+,m_\pm)(1-2s\Sigma^3)\widetilde{\psi}_+(x^+,k^+,n_\pm)|\Omega\rangle \\
&\qquad = \left[1-\mathrm{Prob}(k^+,n_-,s)\right]2\pi\delta(k^+-q^+)\delta_{m_\pm,n_\pm}
\end{aligned}
$$

- Recurring feature of VEVs of fermion bilinears

$$
\begin{aligned}
&\langle\Omega|\psi_\pm^\dagger(x^+,x^-)\psi_\pm(y^+,y^-)|\Omega\rangle \longrightarrow \\
&\qquad \langle\Omega|\psi_+^\dagger(0,x^-)\psi_+(0,y^-)|\Omega\rangle \\
&\qquad \langle\Omega|\psi_+^\dagger(0,x^-)\psi_-(y^+,-L)|\Omega\rangle \to 0 \text{ as } L\to\infty \\
&\qquad \langle\Omega|\psi_-^\dagger(x^+,-L)\psi_+(0,y^-)|\Omega\rangle \to 0 \text{ as } L\to\infty \\
&\qquad \langle\Omega|\psi_-^\dagger(x^+,-L)\psi_-(y^+,-L)|\Omega\rangle
\end{aligned}
$$

- Result for the probability

$$
\begin{aligned}
\mathrm{Prob}(k^+,n_-,s) &= e^{-2\pi\lambda(k^+,n^-,s)} \\
\lambda(k^+,n_-,s) &\equiv \frac{\left[\frac{1}{2}m^2+(2n_-+1-2s)\beta\right]}{|eE(X(k^+))|}
\end{aligned}
$$

- Vector currents (calculated using harmonic oscillator basis)

$$J^{\pm} = \frac{e}{\sqrt{2}} \left(\psi_{\pm}^{\dagger} \psi_{\pm} - \mathrm{Tr}[\psi_{\pm} \psi_{\pm}^{\dagger}] \right)$$

- Regulate by point splitting $J^{\pm}$ in $x^{\mp}$ and $x^{\perp}$

$$\begin{aligned} & J^{+}(x^{+}; x^{-}, y^{-}; x^{\perp}, y^{\perp}) \\ & \quad = \frac{e}{\sqrt{2}} e^{-ie \int_{y^{+}}^{x^{+}} d\xi \cdot A(x(\xi))} \Bigg(\psi_{+}^{\dagger}(x^{+}, x^{-}, x^{\perp}) \psi_{+}(x^{+}, y^{-}, y^{\perp}) \\ & \quad - \mathrm{Tr}\left[\psi_{+}(x^{+}, y^{-}, y^{\perp}) \psi_{+}^{\dagger}(x^{+}, x^{-}, x^{\perp}) \right] \Bigg) \\ & J^{\pm}(x) = \lim_{y \to x} \frac{1}{2} \left(J^{\pm}(x; y) + J^{\pm}(y; x) \right) \end{aligned}$$

- Result for $J^{\pm}$

$$\begin{aligned} & \lim_{L \to \infty} \langle \Omega | J^{+}(x^{+}, x^{-}, x^{\perp}) | \Omega \rangle \\ & \quad = \frac{e^2 B}{4\pi^2} \int_0^{eA_-} dp^{+} e^{-\frac{\pi m^2}{|eE|}} \coth\left[\frac{\pi B}{E(X(p^{+}))} \right] \\ & \lim_{L \to \infty} \langle \Omega | J^{-}(x^{+}, x^{-}, x^{\perp}) | \Omega \rangle_{\mathrm{ren}} \\ & \quad = \frac{e^3 B E(x^{+})}{4\pi^2} (x^{-} + L) e^{-\frac{\pi m^2}{|eE(x^{+})|}} \coth\left[\frac{\pi B}{E(x^{+})} \right] + \dots \end{aligned}$$

- Correspondence limit $B \to 0$

$$\begin{aligned}\langle J^+(x^+, x^-, x^\perp)\rangle &\to -\frac{e^3}{4\pi^3}\int_0^{x^+} du[E(u)]^2 e^{-\frac{\pi m^2}{|eE(u)|}} \\ \langle J^-(x^+, x^-, x^\perp)\rangle_{\rm ren} &\to \frac{e^3[E(x^+)]^2}{4\pi^3}(x^- + L)e^{-\frac{\pi m^2}{|eE(x^+)|}}\end{aligned}$$

- The interesting limit $B \to \infty$

J^+, J^- diverge linearly (phenomenological implications?)

- Axial vector and pseudoscalar (using kernel representation)

$$
\begin{aligned}
J_5^{\pm} &= \sqrt{2}\psi_{\pm}^{\dagger}\gamma_5\gamma_{\pm} \\
J_5 &= \frac{1}{\sqrt{2}}\left(\psi_{+}^{\dagger}\gamma^{-}\gamma_5 + \psi_{-}^{\dagger}\gamma^{+}\gamma_5\psi_{+}\right)
\end{aligned}
$$

$$
\begin{aligned}
\lim_{L\to\infty}\langle\Omega|J_5^{+}(x^{+},x^{-},x^{\perp})|\Omega\rangle &= -\frac{eB}{4\pi^2}\int_0^{eA_-} dp^{+}e^{-\frac{\pi m^2}{|eE|}} \\
\lim_{L\to\infty}\langle\Omega|J_5^{-}(x^{+},x^{-},x^{\perp})|\Omega\rangle &= \frac{e^2E(x^{+})B}{4\pi^2}(x^{-}+L)e^{-\frac{\pi m^2}{|eE|}} \\
\lim_{L\to\infty}\langle\Omega|J_5(x^{+},x^{-},x^{\perp})|\Omega\rangle &= \frac{ie^2E(x^{+})B}{4\pi^2 m}\left[1-e^{-\frac{\pi m^2}{|eE|}}\right]
\end{aligned}
$$

- Axial anomaly

$$
\begin{aligned}
\partial_\mu J_5^{\mu} - 2imJ_5 &= \frac{e^2}{16\pi^2}\epsilon^{\alpha\beta\rho\sigma}F_{\alpha\beta}F_{\rho\sigma} \\
\partial_{+}J^{+} + \partial_{-}J^{-} - 2imJ_5 &= \frac{e^2E(x^{+})B}{2\pi^2}
\end{aligned}
$$

- Simple examples of how crucial a role our initial value surfaces play

$$\lim_{\substack{L\to\infty \\ m\to 0}} \langle\Omega|\partial_+ J_5^+ + \partial_- J_5^-|\Omega\rangle_{++} = \frac{e^2 E(x^+) B}{4\pi^2}$$

(loss of the axial anomaly)

$$-\frac{e}{2\pi}\sum_{n_\pm, s} W^*_{n_\pm}(x^\perp) W_{n_\pm}(x^\perp + \Delta^\perp)$$

$$\times \int_0^{eA_-} dp^+ \left[1 + e^{-2\pi\lambda(p^+, n_-, s)}\right] \cos[(p^+ - eA_-)\Delta_-]$$

(loss of current conservation and renormalizability)

- Neglecting the initial value operator on $x^- = -L$ leads to the explicit loss of unitarity